# Unit - V

# INTEGRAL CALCULUS

## 5.1 Introduction

We are familar with various methods of integration, definite integrals and the associated application of finding the area under a curve.

In this unit we first discuss the topic **Differentiation under the integral sign**, by which we can evaluate certain definite integrals which are either difficult or impossible by known methods of integration.

Later we discuss three **Reduction Formulae** which will be useful in discussing some more **applications of integral calculus : perimeter, surface area and volume** in respect of certain standard curves

## 5.2 Differentiation under the integral sign

This topic deals with the technique of evaluating a definite integral of a function of an independent variable along with a parameter by applying well established rule known as a *Leibnitz rule*. It is important to note that the definite integrals are either difficult or impossible to evaluate by various known methods of integration. Further, starting from the value of one definite integral, applying the rule, we can find the value of an other definite integral which is otherwise difficult/impossible to evaluate.

### 5.21 Leibnitz rule for differentiation under the integral sign

If $f(x, \alpha)$, $\alpha$ being the parameter and $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ are continuous functions then it is obvious that $\int_a^b f(x, \alpha)\, dx$ is a function of $\alpha$, be denoted by $\phi(\alpha)$. That is to say that if

$$\phi(\alpha) = \int_a^b f(x, \alpha)\, dx$$

then Leibnitz rule states that

$$\phi'(\alpha) = \frac{d\phi}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha}\left[f(x, \alpha)\right] dx \quad \text{where } a \text{ and } b \text{ are constants.}$$

In other words the rule means that

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial}{\partial\alpha}[f(x,\alpha)]\,dx$$

## Working procedure for problems

- The given definite integral be denoted by $\phi(\alpha)$, $\alpha$ being the parameter, $x$ is the variable of integration.
- We find $\phi'(\alpha)$ by applying the rule. That is to differentiate the integrand partially with respect to the parameter within the integral sign.
- We integrate between the given limits to obtain a function of $\alpha$. That is, if $\phi'(\alpha) = F(\alpha)$ (*say*) then $\phi(\alpha) = \int F(\alpha)\,d\alpha + c$, $c$ being the constant of integration.
- We evaluate $c$ by taking a suitable value for $\alpha$, with the result we obtain the required $\phi(\alpha)$.

>> Let $\phi(\alpha) = \int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx$ ... (1)

We have by Leibnitz rule,

$$\phi'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}\left[e^{-\alpha x}\cdot\frac{\sin x}{x}\right] = \int_0^\infty e^{-\alpha x}(-x)\frac{\sin x}{x}\,dx$$

Using $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$ where $a = -\alpha$, $b = 1$,

$$\phi'(\alpha) = -\left[\frac{e^{-\alpha x}}{\alpha^2+1}(-\alpha\sin x - \cos x)\right]_{x=0}^{\infty}$$

$$= \frac{1}{\alpha^2+1}\left[e^{-\alpha x}(\alpha\sin x + \cos x)\right]_{x=0}^{\infty}$$

*i.e.,* $\phi'(\alpha) = \dfrac{1}{\alpha^2+1}\left\{0 - e^0(\alpha\sin 0 + \cos 0)\right\} \quad \because \ e^{-\alpha x} \to 0 \text{ as } x \to \infty.$

*i.e.,* $\phi'(\alpha) = \dfrac{-1}{\alpha^2+1}$

$\therefore \quad \phi(\alpha) = -\displaystyle\int \frac{1}{\alpha^2+1}\, d\alpha + c = -\tan^{-1}\alpha + c$

Thus $\phi(\alpha) = -\tan^{-1}\alpha + c$ . . . (2)

Now to find $c$ let us put $\alpha = \infty$ in (2) so that we have

$$\phi(\infty) = -\tan^{-1}(\infty) + c$$

Using (1) in L.H.S we have,

$$\int_0^\infty 0 \cdot \frac{\sin x}{x}\, dx = -\frac{\pi}{2} + c \qquad \because \quad e^{-\alpha x} \to 0 \text{ as } \alpha \to \infty$$

*i.e.,* $0 = -\pi/2 + c \qquad \therefore \ c = \pi/2$

Substituting in (2) the required $\phi(\alpha) = -\tan^{-1}\alpha + (\pi/2)$

Thus $\displaystyle\int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\, dx = \frac{\pi}{2} - \tan^{-1}\alpha = \cot^{-1}\alpha$

Now putting $\alpha = 0$, $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2} - \tan^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Thus $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

>> Let $\phi(\alpha) = \displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$ . . . (i)

$$\therefore \quad \phi'(\alpha) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{x^{\alpha}-1}{\log x}\right) dx, \text{ by Leibnitz rule.}$$

$$= \int_0^1 \frac{1}{\log x}\, x^{\alpha} \log x \, dx \qquad \because \frac{d}{dx}(a^x) = a^x \log a$$

$$i.e., \quad \phi'(\alpha) = \int_0^1 x^{\alpha}\, dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

We have $\phi'(\alpha) = \dfrac{1}{\alpha+1}$

$$\therefore \quad \phi(\alpha) = \int \frac{1}{\alpha+1}\, d\alpha + c$$

$$i.e., \quad \phi(\alpha) = \log(\alpha+1) + c \qquad \ldots (2)$$

Putting $\alpha = 0$, we have $\phi(0) = \log 1 + c$

But $\phi(0) = 0$ from (1) and $\log 1 = 0 \quad \therefore \; c = 0$

Hence we have from (2), $\phi(\alpha) = \log(\alpha+1)$

Thus $\displaystyle\int_0^1 \frac{x^{\alpha}-1}{\log x}\, dx = \log(\alpha+1)$.

Now putting $\alpha = 3$ we get

$$\int_0^1 \frac{x^3-1}{\log x}\, dx = \log 4 \quad \text{or} \quad \int_0^1 \frac{x^3-1}{\log x}\, dx = \log 2^2 = 2\log 2$$

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3. *Evaluate* $\displaystyle\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$ *by differentiating under the integral sign and hence find*

$$\int_0^{\infty} \frac{\tan^{-1} 3x}{x(1+x^2)}\, dx$$

>> Let $\phi(a) = \displaystyle\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$ ... (1)

$$\therefore \quad \phi'(a) = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\tan^{-1} ax}{x(1+x^2)}\right] dx, \text{ by Leibnitz rule.}$$

i.e., $$\phi'(a) = \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{x}{1+(ax)^2}\, dx$$

i.e., $$\phi'(a) = \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)} \quad \ldots (2)$$

We shall integrate R.H.S by resolving into partial fractions taking $x^2 = t$, only for convenience.

Let $$\frac{1}{(1+t)(1+a^2t)} = \frac{A}{1+t} + \frac{B}{1+a^2t}$$

or $$1 = A(1+a^2t) + B(1+t)$$

By putting $t = -1$ we get $A = \frac{1}{1-a^2}$.

Also by putting $t = -1/a^2$ we get $B = \frac{-a^2}{1-a^2}$

Hence $$\frac{1}{(1+t)(1+a^2t)} = \frac{1}{1-a^2}\frac{1}{1+t} - \frac{a^2}{1-a^2}\frac{1}{1+a^2t}$$

Replacing back $t = x^2$ and integrating w.r.t $x$ betwee 0 and $\infty$ we have,

$$\int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)} = \frac{1}{1-a^2}\int_0^\infty \frac{dx}{1+x^2} - \frac{a^2}{1-a^2}\int_0^\infty \frac{dx}{1+(ax)^2}$$

i.e., $$\phi'(a) = \frac{1}{1-a^2}\left\{[\tan^{-1}x]_0^\infty - a^2\cdot\frac{1}{a}[\tan^{-1}ax]_o^\infty\right\}$$

i.e., $$\phi'(a) = \frac{1}{1-a^2}\left\{\frac{\pi}{2} - a\frac{\pi}{2}\right\} = \frac{\pi(1-a)}{2(1-a^2)} = \frac{\pi}{2(1+a)}$$

$\therefore$ $$\phi(a) = \frac{\pi}{2}\int \frac{da}{1+a} + c$$

i.e., $$\phi(a) = \frac{\pi}{2}\log(1+a) + c \quad \ldots (3)$$

To evaluate $c$, let us put $a = 0$ in (3)

Hence $\phi(0) = \frac{\pi}{2}\log 1 + c$

But $\phi(0) = 0$ from (1) and $\log 1 = 0$ $\therefore$ $c = 0$

Hence we have from (3), $\phi(a) = \frac{\pi}{2}\log(1+a)$

Thus $\displaystyle\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a)$

Now by putting $a = 3$ we get,

$$\int_0^\infty \frac{\tan^{-1} 3x}{x(1+x^2)}\,dx = \frac{\pi}{2}\log 4 = \frac{\pi}{2}\log 2^2 = \frac{\pi}{2}\cdot 2\log 2$$

Thus $\displaystyle\int_0^\infty \frac{\tan^{-1} 3x}{x(1+x^2)}\,dx = \pi\log 2$

>> Let $\displaystyle\phi(\alpha) = \int_0^\infty e^{-x^2}\cos\alpha x\,dx$ ...(1)

$\therefore$ $\displaystyle\phi'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}(e^{-x^2}\cos\alpha x)\,dx$, by Leibnitz rule.

*i.e.,* $\displaystyle\phi'(\alpha) = \int_0^\infty e^{-x^2}(-\sin\alpha x\cdot x)\,dx$

*i.e.,* $\displaystyle\phi'(\alpha) = \int_0^\infty \sin\alpha x\,(-x\,e^{-x^2})\,dx$

Noting that $\int -x\,e^{-x^2}\,dx = \frac{1}{2}e^{-x^2}$, we have on integration by parts,

$$\phi'(\alpha) = \left[\sin\alpha x\cdot\frac{1}{2}e^{-x^2}\right]_{x=0}^{\infty} - \int_0^\infty \frac{1}{2}e^{-x^2}\cdot\cos\alpha x\cdot\alpha\,dx$$

*i.e.,* $\displaystyle\phi'(\alpha) = 0 - \frac{\alpha}{2}\int_0^\infty e^{-x^2}\cos\alpha x\,dx$ or $\phi'(\alpha) = -\frac{\alpha}{2}\phi(\alpha)$ by using (1)

*[Here we adopt a different technique to find $\phi(\alpha)$]*

$$\frac{\phi'(\alpha)}{\phi(\alpha)} = -\frac{\alpha}{2} \text{ and on integration we have}$$

*i.e.,* $$\int \frac{\phi'(\alpha)}{\phi(\alpha)}\, d\alpha = -\int \frac{\alpha}{2}\, d\alpha + c$$

*i.e.,* $$\log \phi(\alpha) = -\frac{\alpha^2}{4} + c$$

$$\Rightarrow \quad \phi(\alpha) = e^{-\alpha^2/4 + c} \quad \text{or} \quad \phi(\alpha) = e^c\, e^{-\alpha^2/4}$$

Putting $\alpha = 0$ we have,

$$\phi(0) = e^c \cdot 1$$

$$\int_0^\infty e^{-x^2} \cdot 1\, dx = e^c$$

**Note** : $\int_0^\infty e^{-x^2} dx$ *is to be evaluated using gamma functions and it works out to be $\sqrt{\pi}/2$ which is to be assumed here.*

Hence we have $\sqrt{\pi}/2 = e^c$, with the result we write,

$$\phi(\alpha) = \sqrt{\pi}/2 \cdot e^{-\alpha^2/4}$$

Thus $$\int_0^\infty e^{-x^2} \cos \alpha x\, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$$

>> Let $$\phi(a) = \int_0^\pi \frac{\log(1 + a\cos x)}{\cos x}\, dx \qquad \ldots (1)$$

$$\therefore \quad \phi'(a) = \int_0^\pi \frac{1}{\cos x} \cdot \frac{1}{1 + a\cos x} \cdot \cos x\, dx, \text{ by Leibnitz rule.}$$

$$\phi'(a) = \int_0^\pi \frac{dx}{1 + a\cos x}$$

We employ a known method to integrate R.H.S by using the substitution $\tan(x/2) = t$. This will give us,

$$\cos x = \frac{1-t^2}{1+t^2} \text{ and } dx = \frac{2dt}{1+t^2}\ ;\ t \text{ varies from } 0 \text{ to } \infty$$

Hence $\phi'(a) = \int_0^\infty \dfrac{2\,dt/1+t^2}{1+a\cdot\dfrac{1-t^2}{1+t^2}} = 2\int_0^\infty \dfrac{dt}{(1+t^2)+a(1-t^2)}$

*i.e.,* $\phi'(a) = 2\int_0^\infty \dfrac{dt}{(1+a)+(1-a)t^2} = \dfrac{2}{1-a}\int_0^\infty \dfrac{dt}{t^2+\left(\dfrac{1+a}{1-a}\right)}$

Denoting $b^2 = \dfrac{1+a}{1-a}$ we have,

$$\phi'(a) = \frac{2}{1-a}\int_0^\infty \frac{dt}{t^2+b^2}$$

*ie.,* $= \dfrac{2}{1-a}\left[\dfrac{1}{b}\tan^{-1}\left(\dfrac{t}{b}\right)\right]_{t=0}^{\infty} = \dfrac{2}{(1-a)b}\left\{\tan^{-1}(\infty) - \tan^{-1}(0)\right\}$

*ie.,* $\phi'(a) = \dfrac{2}{(1-a)}\,\dfrac{\sqrt{1-a}}{\sqrt{1+a}}\,\dfrac{\pi}{2}$

*i.e.,* $\phi'(a) = \dfrac{\pi}{\sqrt{1-a^2}}$ $\left(\textit{Note}: \int_0^\infty \dfrac{dx}{1+a\cos x} = \dfrac{\pi}{\sqrt{1-a^2}},\ a<1\right)$

$\therefore$ $\phi(a) = \pi\int \dfrac{da}{\sqrt{1-a^2}} + c$

*i.e.,* $\phi(a) = \pi\sin^{-1}a + c$

Putting $a = 0$ we have, $\phi(0) = 0 + c$

But $\phi(0) = 0$ from (1) and we have $c = 0$

Hence we get $\phi(a) = \pi\sin^{-1}a$

Thus we have proved that,

$$\int_0^\pi \frac{\log(1+a\cos x)}{\cos x}\,dx = \pi\sin^{-1}a$$

>> Let $\phi(a) = \int_0^{\pi} \log(1 + a\cos x)\,dx$ ...(1)

$\therefore \quad \phi'(a) = \int_0^{\pi} \frac{\cos x}{1 + a\cos x}\,dx$, by Leibnitz rule.

$$= \frac{1}{a}\int_0^{\pi} \frac{a\cos x}{1 + a\cos x}\,dx = \frac{1}{a}\int_0^{\pi} \frac{(1 + a\cos x - 1)}{1 + a\cos x}\,dx$$

*i.e.,* $\phi'(a) = \frac{1}{a}\int_0^{\pi} dx - \frac{1}{a}\int_0^{\pi} \frac{dx}{1 + a\cos x}$

*i.e.,* $\phi'(a) = \frac{\pi}{a} - \frac{1}{a}\cdot\frac{\pi}{\sqrt{1 - a^2}}$ (*Refer previous example*)

$\therefore \quad \phi(a) = \pi\left[\int \frac{da}{a} - \int \frac{da}{a\sqrt{1 - a^2}}\right] + c$

Using the formula,

$$\int \frac{dx}{x\sqrt{1 - x^2}} = -\operatorname{sec}h^{-1} x = -\log\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) + c$$

$$\phi(a) = \pi\left[\log a + \log\left(\frac{1 + \sqrt{1 - a^2}}{a}\right)\right]$$

or $\phi(a) = \pi\log(1 + \sqrt{1 - a^2}) + c$

Putting $a = 0$ we have, $\phi(0) = \pi\log 2 + c$

But $\phi(0) = 0$ from (1) and we have $c = -\pi\log 2$

Hence we get $\phi(a) = \pi\log(1 + \sqrt{1 - a^2}) - \pi\log 2$

**Thus we have** $\int_0^{\pi} \log(1 + a\cos x)\,dx = \pi\log\left[\frac{1 + \sqrt{1 - a^2}}{2}\right], \quad a < 1$

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>> Let $\phi(y) = \int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\, dx$ ... (1)

$\therefore \quad \phi'(y) = \int_0^{\pi/2} \frac{1}{\sin^2 x} \cdot \frac{1}{1+y\sin^2 x} \cdot \sin^2 x\, dx$, by Leibnitz rule

*i.e.,* $\quad \phi'(y) = \int_0^{\pi/2} \frac{dx}{1+y\sin^2 x}$

(*We employ a known method to integrate R.H.S*)

$$\phi'(y) = \int_0^{\pi/2} \frac{dx}{\cos^2 x + (1+y)\sin^2 x} = \int_0^{\pi/2} \frac{dx}{\cos^2 x\,[1+(1+y)\tan^2 x]}$$

*i.e.,* $\quad \phi'(y) = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{1+(1+y)\tan^2 x}$

Taking $t = \sqrt{1+y}\tan x$ we get $dt = \sqrt{1+y}\sec^2 x\, dx$

Also $t$ varis from 0 to $\infty$.

Hence $\phi'(y) = \int_0^{\infty} \frac{dt/\sqrt{1+y}}{1+t^2}$

*i.e.,* $\quad \phi'(y) = \frac{1}{\sqrt{1+y}}[\tan^{-1} t]_0^{\infty} = \frac{1}{\sqrt{1+y}}[\tan^{-1}(\infty) - \tan^{-1}(0)]$

*i.e.,* $\quad \phi'(y) = \frac{\pi}{2\sqrt{1+y}}$

$\therefore \quad \phi(y) = \pi \int \frac{dy}{2\sqrt{1+y}} + c$

*i.e.,* $\quad \phi(y) = \pi\sqrt{1+y} + c$

Putting $y = 0$ we have $\phi(0) = \pi + c$

But $\quad \phi(0) = 0$ from (1) and hence $c = -\pi$

Hence we get $\phi(y) = \pi\sqrt{1+y} - \pi$

Thus we have $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\, dx = \pi[\sqrt{1+y} - 1]$

>> Let $\phi(\alpha) = \int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}$

Differentiate *w.r.t.* $\alpha$

Now $\phi'(\alpha) = \int_0^{\pi} \frac{\partial}{\partial \alpha}\left[\frac{1}{\alpha - \cos x}\right] dx = \pi \frac{d}{d\alpha}(\alpha^2 - 1)^{-1/2}$

where Leibnitz rule is employed for differentiating under the integral sign.

*i.e.,* $\int_0^{\pi} \frac{-1}{(\alpha - \cos x)^2} dx = \pi \cdot -\frac{1}{2}(\alpha^2 - 1)^{-3/2} \cdot 2\alpha$

or $\int_0^{\pi} \frac{dx}{(\alpha - \cos x)^2} = \frac{\pi \alpha}{(\alpha^2 - 1)^{3/2}}$

Now by putting $\alpha = 2$ we get,

$$\int_0^{\pi} \frac{dx}{(2 - \cos x)^2} = \frac{2\pi}{3\sqrt{3}}$$

**Remark** : *Here it may be observed that starting from the value of one integral, which infact can be obtained comfortably, we have found the value of an other integral easily (which is otherwise difficult to evaluate) by applying Leibnitz rule.*

>> Let $\phi(m) = \int_0^1 x^m \, dx = \frac{1}{m+1}$.

Differentiate w.r.t $m$, where We shall apply Leibnitz rule to differentiate under the integral sign.

$$\therefore \quad \int_0^1 \frac{\partial}{\partial m}(x^m)\,dx = \frac{d}{dm}\left(\frac{1}{m+1}\right)$$

$$\text{i.e.,} \quad \int_0^1 x^m (\log x)\,dx = \frac{-1}{(m+1)^2}$$

Applying the rule again we have,

$$\int_0^1 x^m \log x\,(\log x)\,dx = (-1)(-2)(m+1)^{-3} = (-1)^2\,2!\,(m+1)^{-3}$$

$$\text{i.e.,} \quad \int_0^1 x^m (\log x)^2\,dx = (-1)^2\,2!\,(m+1)^{-3}$$

Applying the rule once again we have

$$\int_0^1 x^m \log x\,(\log x)^2\,dx = (-1)^2\,2!\,(-3)(m+1)^{-4}$$

$$\text{i.e.,} \quad \int_0^1 x^m (\log x)^3\,dx = (-1)^3\,3!\,(m+1)^{-4}$$

Continuing like this, by differentiating $n$ times we get

$$\int_0^1 x^m (\log x)^n\,dx = (-1)^n\,n!\,(m+1)^{-(n+1)}$$

$$\text{Thus} \quad \int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n\,n!}{(m+1)^{n+1}}$$

>> Let $\phi(a) = \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\pi/a} = \frac{\sqrt{\pi}}{2} a^{-1/2}$

$\therefore \quad \phi'(a) = \int_0^\infty e^{-ax^2}(-x^2)\,dx = \frac{\sqrt{\pi}}{2} \cdot -\frac{1}{2} a^{-3/2}$

where we have employed Leibnitz rule to differentiate under the integral sign.

Differentiating again *w.r.t* $a$ we have,

$$\phi''(a) = \int_0^\infty e^{-ax^2}(-x^2)^2\,dx = \frac{\sqrt{\pi}}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} a^{-5/2}$$

Similarly we have,

$$\phi'''(a) = \int_0^\infty e^{-ax^2}(-x^2)^3\,dx = \frac{\sqrt{\pi}}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2} a^{-7/2}$$

Continuing like this, on differentiating $n$ times we have,

$$\phi^{(n)}(a) = \int_0^\infty e^{-ax^2}(-x^2)^n\,dx = \frac{\sqrt{\pi}}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2} \cdots -\frac{(2n-1)}{2} a^{-(2n+1)/2}$$

or $\quad \int_0^\infty e^{-ax^2}(x^2)^n(-1)^n\,dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n} (-1)^n a^{-\left(n+\frac{1}{2}\right)}$

Thus $\quad \int_0^\infty e^{-ax^2} x^{2n}\,dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n a^{n+\frac{1}{2}}}$

*Evaluate the following integrals by differentiating under the integral sign [1 to 3]*

1. $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx, \quad a > -1$

2. $\int_0^\infty x\, e^{-x^2} \sin ax\, dx$

3. $\int_0^\pi \frac{\log(1+\sin a \cos x)}{\cos x}\,dx$

4. Differentiating $\int_0^x \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ under the integral sign, evaluate

$$\int_0^x \frac{dx}{(x^2+a^2)^2}$$

5. Given that $\int_0^\infty \frac{dx}{x^2+a} = \frac{\pi}{2\sqrt{a}}, a > 0$ show that

$$\int_0^\infty \frac{dx}{(x^2+a)^{n+1}} = \frac{\pi}{2} \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!\, a^{n+\frac{1}{2}}}$$

6. If $\phi(y) = \int_0^\infty e^{-x^2}\sin 2yx\,dx$, show that $\frac{d\phi(y)}{dy} + 2y\,\phi(y) = 1$

1. $\log(1+a)$

2. $\frac{\sqrt{\pi}\,a}{4}\, e^{-a^2/4}$

3. $\pi a$

4. $\frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}(x/a)$

Reduction formulae is basically a recurrence relation which reduces integral of functions of higher degree in the form $\int [f(x)]^n\, dx$, $\int [f(x)]^m\, [g(x)]^n\, dx$ (*where m and n are non negative integers*) to lower degree. The successive application of the recurrence relation finally end up with a function of degree 0 or 1 so that we can easily complete the integration process.

We discuss three standard reduction formulae in the form of indefinite integrals and the evaluation of them with standard limits of integration.

Let $I_n = \int \sin^n x \; dx$

$= \int \sin^{n-1} x \cdot \sin x \; dx = \int u\,v \; dx$ (say) .

We have the rule of integration by parts,

$$\int uv \; dx = u \int v \; dx - \int \int v \; dx \cdot u' \; dx$$

$\therefore \quad I_n = \sin^{n-1} x \; (-\cos x) - \int (-\cos x)\; (n-1)\; \sin^{n-2} x \cdot \cos x \; dx$

$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x \; dx$

$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \,(1 - \sin^2 x)\; dx$

$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \; dx - (n-1) \int \sin^n x \; dx$

ie., $\quad I_n = -\sin^{n-1} x \cdot \cos x + (n-1)\, I_{n-2} - (n-1)\, I_n$

ie., $\quad I_n \,[\,1 + (n-1)\,] = -\sin^{n-1} x \cdot \cos x + (n-1)\, I_{n-2}$

$$\therefore \quad I_n = \int \sin^n x \; dx = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n}\, I_{n-2} \qquad \ldots (1)$$

This is the required reduction formula.

(i) *To find* $\int \sin^4 x \; dx$

>> Comparing with the L.H.S. of (1), we need to take $n = 4$ and use the established result.

$$\therefore \quad I_4 = \int \sin^4 x \; dx = \frac{-\sin^3 x \; \cos x}{4} + \frac{3}{4}\, I_2$$

We need to apply the result (1) again by taking $n = 2$

ie., $$I_4 = \frac{-\sin^3 x \; \cos x}{4} + \frac{3}{4} \left\{ \frac{-\sin x \; \cos x}{2} + \frac{1}{2}\, I_0 \right\}$$

We cannot find $I_0$ from (1). But basically we have

$$I_0 = \int \sin^0 x \; dx = \int 1 \; dx = x$$

Thus $I_4 = \int \sin^4 x \; dx = \dfrac{-\sin^3 x \cos x}{4} - \dfrac{3}{8} \sin x \cos x + \dfrac{3x}{8} + c$

(ii) *To find* $\int \sin^5 x \; dx$

>> $I_5 = \int \sin^5 x \; dx = \dfrac{-\sin^4 x \cos x}{5} + \dfrac{4}{5} I_3$

ie., $= \dfrac{-\sin^4 x \cos x}{5} + \dfrac{4}{5} \left\{ \dfrac{-\sin^2 x \cos x}{3} + \dfrac{2}{3} I_1 \right\}$

But $I_1 = \int \sin^1 x \; dx = \int \sin x \; dx = -\cos x$

$\therefore \quad \int \sin^5 x \; dx = \dfrac{-\sin^4 x \cos x}{5} - \dfrac{4 \sin^2 x \cos x}{15} - \dfrac{8}{15} \cos x + c$

**Corollary** : Evaluation of $\int_0^{\pi/2} \sin^n x \; dx$

Let $I_n = \int_0^{\pi/2} \sin^n x \; dx$

*Equation* (1) *must be established first.*

$\therefore \quad$ from (1) $I_n = -\left[ \dfrac{\sin^{n-1} x \cos x}{n} \right]_0^{\pi/2} + \dfrac{n-1}{n} I_{n-2}$

But $\cos(\pi/2) = 0 = \sin 0.$

Thus $I_n = \dfrac{n-1}{n} I_{n-2}$ . . . (2)

We use this recurrence relation to find $I_{n-2}$ by simply replacing $n$ by $(n-2)$.

ie., $I_{n-2} = \dfrac{n-3}{n-2} I_{n-4}$

Hence $I_n = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} I_{n-4}$, by back substitution.

Similarly from (2) $I_{n-4} = \dfrac{n-5}{n-4} I_{n-6}$

Hence $I_n = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} I_{n-6}$, again by back substitution.

Continuing like this, the reduction process will end up as follows.

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} I_1 \quad \text{if } n \text{ is odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} I_0 \quad \text{if } n \text{ is even}$$

But $\quad I_1 = \int_0^{\pi/2} \sin x \, dx = -\left[\cos x\right]_0^{\pi/2} = -(0-1) = 1$

and $\quad I_0 = \int_0^{\pi/2} \sin^0 x \, dx = \int_0^{\pi/2} dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$

Thus we have,

$$\int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

**Illustrations** : *To find* (i) $\int_0^{\pi/2} \sin^4 x \, dx$ (ii) $\int_0^{\pi/2} \sin^5 x \, dx$

(i) $\quad \int_0^{\pi/2} \sin^4 x \, dx = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$

(ii) $\quad \int_0^{\pi/2} \sin^5 x \, dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$

[illegible]

Let $\quad I_n = \int \cos^n x \, dx$

$$= \int \cos^{n-1} x \cdot \cos x \, dx$$

Integrating by parts we have,

$$I_n = \cos^{n-1} x \cdot \sin x - \int \sin x \, (n-1) \cos^{n-2} x \, (-\sin x) \, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x \, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x)\, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$I_n = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} - (n-1) I_n$$

*ie.,* $I_n \, [1 + (n-1)] = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2}$

$\therefore \quad I_n = \int \cos^n x \, dx = \dfrac{\cos^{n-1} x \cdot \sin x}{n} + \dfrac{n-1}{n} I_{n-2}$ ...(1)

Next, let $I_n = \int_0^{\pi/2} \cos^n x \, dx$

$\therefore$ from (1), $I_n = \left[ \dfrac{\cos^{n-1} x \cdot \sin x}{n} \right]_0^{\pi/2} + \dfrac{n-1}{n} I_{n-2}$

But $\cos(\pi/2) = 0 = \sin 0$.

Thus $I_n = \dfrac{n-1}{n} I_{n-2}$ ...(2)

**Remark**: *This result is same as (2) of 5.31 and proceeding on the same lines as in 5.31 we can obtain the result which is identically same as that of* $\int_0^{\pi/2} \sin^n x \, dx$.

*Infact we can easily conclude that* $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$ *because of a property of definite integrals,* $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ *as we have* $f(x) = \sin^n x$ ; $f(\pi/2 - x) = \{\sin(\pi/2 - x)\}^n = (\cos x)^n = \cos^n x$

$$\therefore \quad \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

(i) *To find* $\int \cos^6 x\,dx$

>> We have from (1) by putting $n = 6$,

$$I_6 = \int \cos^6 x\,dx = \frac{\cos^5 x \sin x}{6} + \frac{5}{6} I_4$$

*i.e.,*
$$= \frac{\cos^5 x \sin x}{6} + \frac{5}{6}\left\{\frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2\right\}$$

$$= \frac{\cos^5 x \sin x}{6} + \frac{5}{24}\cos^3 x \sin x + \frac{5}{8}\left\{\frac{\cos x \sin x}{2} + \frac{1}{2} I_0\right\}$$

But $\quad I_0 = \int \cos^0 x\,dx = \int 1\,dx = x$

$$\therefore \quad \int \cos^6 x\,dx = \frac{\cos^5 x \sin x}{6} + \frac{5\cos^3 x \sin x}{24} + \frac{5\cos x \sin x}{16} + \frac{5x}{16} + c$$

*To find* (ii) $\int_0^{\pi/2} \cos^6 x\,dx$ (iii) $\int_0^{\pi/2} \cos^7 x\,dx$

(ii) $$\int_0^{\pi/2} \cos^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \mathbf{\frac{5\pi}{32}}$$

(iii) $$\int_0^{\pi/2} \cos^7 x\,dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \mathbf{\frac{16}{35}}$$

Let $\quad I_{m,n} = \int \sin^m x\,(\cos^n x\,dx)$

$$= \int \sin^{m-1} x\,(\sin x \cos^n x)\,dx = \int uv\ dx \quad (say)$$

We have $\int uv\,dx = u\int v\,dx - \int \int v\,dx \cdot u'\,dx$

Here $\int v\,dx = \int \sin x \cos^n dx$

Put $\quad \cos x = t \quad \therefore \quad -\sin x\,dx = dt$

Hence $\int v\,dx = \int -t^n\,dt = -\dfrac{t^{n+1}}{n+1} = -\dfrac{\cos^{n+1}x}{n+1}$

Now $I_{m,n} = (\sin^{m-1}x)\left(\dfrac{-\cos^{n+1}x}{n+1}\right) - \int \dfrac{-\cos^{n+1}x}{n+1}\,(m-1)\sin^{m-2}x\cos x\,dx$

*ie.,* $= -\dfrac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2}x\cos^{n+2}x\,dx$

$= -\dfrac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2}x\cdot\cos^n x\cdot\cos^2 x\,dx$

$= -\dfrac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,(1-\sin^2 x)\,dx$

$= -\dfrac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,dx - \dfrac{m-1}{n+1}\int \sin^m x\cos^n x\,dx$

$I_{m,n} = \dfrac{-\sin^{m-1}x\cos^{n+1}x}{n+1} + \dfrac{m-1}{n+1}I_{m-2,n} - \dfrac{m-1}{n+1}I_{m,n}$

*ie.,* $I_{m,n}\left[1+\dfrac{m-1}{n+1}\right] = \dfrac{1}{n+1}\left[-\sin^{m-1}x\cos^{n+1}x + (m-1)\,I_{m-2,n}\right]$

$I_{m,n}\left[\dfrac{m+n}{n+1}\right] = \dfrac{1}{n+1}\left[-\sin^{m-1}x\cos^{n+1}x + (m-1)\,I_{m-2,n}\right]$

$\therefore \quad I_{m,n} = \int \sin^m x\cos^n x\,dx = -\dfrac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \dfrac{m-1}{m+n}I_{m-2,n} \qquad \ldots(1)$

**Note** : *If we decompose* $\sin^m x\cos^n x = (\sin^m x\cos x)\cos^{n-1}x$ *and integrate by parts, by taking* $u = \cos^{n-1}x,\ v = \sin^m x\cos x$ *we can obtain*

$$I_{m,n} = \frac{-\sin^{m+1}x\cdot\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}I_{m,n-2} \qquad \ldots(2)$$

Now, let $I_{m,n} = \displaystyle\int_0^{\pi/2} \sin^m x\cos^n x\,dx$

$\therefore$ from (1), $I_{m,n} = -\left[\dfrac{\sin^{m-1}x\cos^{n+1}x}{m+n}\right]_0^{\pi/2} + \dfrac{m-1}{m+n}I_{m-2,n}$

*i.e.,* $I_{m,n} = \dfrac{m-1}{m+n}I_{m-2,n}$ $[\because \cos(\pi/2) = 0 = \sin 0]$

$$\therefore \quad I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}$$

Hence $I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} I_{m-4,n}$ by back substitution.

Continuing like this we obtain

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \times \begin{cases} \frac{2}{3+n} I_{1,n} & \text{if } m \text{ is odd} \\ \frac{1}{2+n} I_{0,n} & \text{if } m \text{ is even} \end{cases}$$

Now $I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x \, dx = -\left[\frac{\cos^{n+1} x}{n+1}\right]_0^{\pi/2} = \frac{1}{n+1}$

and $I_{0,n} = \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$

$$\therefore \quad I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \times \frac{1}{n+1} \quad \text{if } m \text{ is odd and } n \text{ is even or odd.}$$

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n}$$

$$\times \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & \text{if } m \text{ is even and } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } m \text{ is even and } n \text{ is even} \end{cases}$$

**Note** : *This reduction formula for all the cases can be represented as follows.*

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)(m+n-4)\cdots} \times k$$

*where $k = \pi/2$ when m and n are even and $k = 1$ other wise. This is known as* ***Walli's rule.***

Illustrations

(i) $\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx = \frac{[(4)(2)][(3)(1)]}{9 \times 7 \times 5 \times 3 \times 1} = \frac{8}{315}$

(ii) $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx = \frac{[(6)(4)(2)][(4)(2)]}{12 \times 10 \times 8 \times 6 \times 4 \times 2} = \frac{1}{120}$

(iii) $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{[(5)(3)(1)][(4)(2)]}{11 \times 9 \times 7 \times 5 \times 3 \times 1} = \frac{8}{693}$

(iv) $\int_0^{\pi/2} \sin^8 x \cos^6 x \, dx = \frac{[(7)(5)(3)(1)][(5)(3)(1)]}{14 \times 12 \times 10 \times 8 \times 6 \times 4 \times 2} \cdot \frac{\pi}{2} = \frac{5\pi}{4096}$

## Evaluation of definite integrals

**Note :** 1. *The pre requisite for the evaluation of definite integrals are its various properties. In the ultimate step, the evaluation is completed with the help of the reduction formulae.*

2. *In the cases of definite integrals involving algebraic functions like*
   (i) $a - x$, *we can use the substitution* $x = a \sin^2 \theta$
   (ii) $a + x$, *we can use the substitution* $x = a \tan^2 \theta$.

## To remember

(i) $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \times k$

*where* $k = \pi/2$ *only when n is even.*

(ii) $\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots} \times k$

*where* $k = \pi/2$ *only when m and n are even and* $k = 1$, *otherwise.*

**11.** $\int_0^{\pi} \sin^5 (x/2) \, dx$ **12.** $\int_0^{\pi} \sin^4 x \, dx$ **13.** $\int_0^{\pi} x \sin^8 x \, dx$

**14.** $\int_0^{\pi} x \cos^6 x \, dx$ **15.** $\int_{-\pi/2}^{\pi/2} \cos^8 x \, dx$ **16.** $\int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

**17.** $\int_0^{\pi} \sin^6 x \cos^4 x \, dx$

11. Let $I = \int_0^{\pi} \sin^5 (x/2)\, dx$

Put $x/2 = y \quad \therefore \quad dx = 2\, dy$ If $x = 0,\ y = 0$ ; If $x = \pi,\ y = \pi/2$

$$\therefore \quad I = 2 \int_0^{\pi/2} \sin^5 y\, dy$$

$$= 2 \cdot \frac{4}{5} \cdot \frac{2}{3} \text{ by reduction formula.}$$

Thus $I = 16/15$

12. Let $I = \int_0^{\pi} \sin^4 x\, dx$

If $f(x) = \sin^4 x$ and $2a = \pi$ or $a = \pi/2$

$$f(2a - x) = \sin^4(\pi - x) = \sin^4 x = f(x) \quad ie., \ f(2a - x) = f(x)$$

Thus by the property $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$ we have,

$$I = 2 \int_0^{\pi/2} \sin^4 x\, dx = 2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ by reduction formula.}$$

Thus $I = 3\pi/8$

13. Let $I = \int_0^{\pi} x \sin^8 x\, dx$

We have the property $\int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx$

$$\therefore \quad I = \int_0^{\pi} (\pi - x) \sin^8(\pi - x)\, dx = \int_0^{\pi} (\pi - x) \sin^8 x\, dx$$

$$= \pi \int_0^{\pi} \sin^8 x\, dx - \int_0^{\pi} x \sin^8 x\, dx$$

$$I = \pi \int_0^{\pi} \sin^8 x\,dx - I \quad \text{or} \quad 2I = \pi \cdot 2 \int_0^{\pi/2} \sin^8 x\,dx \quad \textit{(As in Example-12)}$$

Hence $I = \pi \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, by reduction formula.

Thus $\quad \mathbf{I = 35\pi^2 / 256}$

---

14. Let $I = \int_0^{\pi} x \cos^6 x\ dx$

$$I = \int_0^{\pi} (\pi - x)\cos^6(\pi - x)\ dx = \int_0^{\pi} (\pi - x)\cos^6 x\ dx$$

$$I = \pi \int_0^{\pi} \cos^6 x\ dx - \int_0^{\pi} x\cos^6 x\ dx = \pi \int_0^{\pi} \cos^6 x\ dx - I$$

$$2I = \pi \cdot 2 \int_0^{\pi/2} \cos^6 x\ dx$$

$\therefore \quad I = \pi \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ by reduction formula.

Thus $\quad \mathbf{I = 5\pi^2 / 32}$

---

15. Let $I = \int_{-\pi/2}^{\pi/2} \cos^8 x\ dx$

We have the property : $\int_{-a}^{a} f(x)\ dx = 2\int_0^{a} f(x)\ dx$ if $f(-x) = f(x)$

Here $\quad \cos^8(-x) = \cos^8 x$ and hence we have,

$$I = 2\int_0^{\pi/2} \cos^8 x\ dx$$

$$= 2 \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ by reduction formula.}$$

Thus $\quad \mathbf{I = 35\pi / 128}$

---

16. Let $I = \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

$$I = \int_0^{\pi} (\pi - x) \sin^2 (\pi - x) \cos^4 (\pi - x) \, dx \text{ , by a property.}$$

$$= \int_0^{\pi} (\pi - x) \sin^2 x \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx - \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx - I$$

$$2I = \pi \cdot 2 \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

$\therefore$ $I = \pi \cdot \dfrac{(1)\cdot(3)\,(1)}{6 \times 4 \times 2} \cdot \dfrac{\pi}{2}$ by reduction formula.

Thus $I = \pi^2 / 32$

---

17. Let $I = \int_0^{\pi} \sin^6 x \cos^4 x$

$$I = 2 \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx \text{ , by a property.}$$

$\therefore$ $I = 2 \cdot \dfrac{[(5)(3)(1)]\,[(3)(1)]}{10 \times 8 \times 6 \times 4 \times 2} \cdot \dfrac{\pi}{2}$ by reduction formula.

Thus $I = 3\pi / 256$

---

18. *Evaluate* $\int_0^{\pi/6} \cos^4 3x \sin^2 6x \, dx$ *using reduction formula.*

>> Let $I = \int_0^{\pi/6} \cos^4 3x \sin^2 6x \, dx$

$\sin 6x = 2\sin 3x \cos 3x \quad \because \sin\theta = 2\sin(\theta/2)\cos(\theta/2)$

$$\therefore \quad I = \int_0^{\pi/6} \cos^4 3x\,(2\sin 3x \cos 3x)^2\,dx$$

*ie.,*

$$I = 4\int_0^{\pi/6} \sin^2 3x \cos^6 3x\,dx$$

Put $3x = y \quad \therefore \; dx = dy/3.$

If $x = 0,\ y = 0$ ; If $x = \pi/6,\ y = \pi/2$

$$\therefore \quad I = 4\int_{y=0}^{\pi/2} \sin^2 y \cos^6 y\,\frac{dy}{3} = \frac{4}{3}\int_0^{\pi/2} \sin^2 y \cos^6 y\,dy$$

$$I = \frac{4}{3}\cdot\left[\frac{(1)\cdot(5)(3)(1)}{8\times 6\times 4\times 2}\cdot\frac{\pi}{2}\right] \text{ by reduction formula.}$$

Thus $I = 5\pi/192$

>> Let $I = \int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\,d\theta$

*ie.,*

$$I = \int_0^{\pi} \frac{[2\sin(\theta/2)\cos(\theta/2)]^4\,d\theta}{[2\cos^2(\theta/2)]^2}$$

$$= \int_0^{\pi} \frac{16\sin^4(\theta/2)\cos^4(\theta/2)}{4\cos^4(\theta/2)}\,d\theta$$

$$I = 4\int_0^{\pi} \sin^4(\theta/2)\,d\theta$$

Put $\theta/2 = \phi \quad \therefore \; d\theta = 2\,d\phi$ and $\phi$ varies from $0$ to $\pi/2$.

$$\therefore \quad I = 4 \int_{\phi=0}^{\pi/2} \sin^4 \phi \cdot 2d\phi$$

*ie.,* $I = 8 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, by reduction formula.

Thus $\mathbf{I = 3\pi / 2}$

>> Let $I = \int_0^2 \frac{x^4}{\sqrt{4-x^2}}\, dx$

Put $x^2 = 4 \sin^2 \theta$ or $x = 2 \sin \theta$ $\therefore$ $dx = 2 \cos \theta \; d\theta$,

$\theta$ varies from 0 to $\pi/2$ and $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$.

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{16 \sin^4 \theta \cdot 2 \cos \theta \; d\theta}{2 \cos \theta} = 16 \int_0^{\pi/2} \sin^4 \theta \; d\theta$$

Hence $I = 16 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, by reduction formula.

Thus $\mathbf{I = 3\pi}$

>> Let $I = \int_0^1 x^2 (1-x^2)^{3/2}\, dx$

Put $x = \sin \theta$ $\therefore$ $dx = \cos \theta \, d\theta$ and $\theta$ varies from 0 to $\pi/2$.

$$(1-x^2)^{3/2} = (\cos^2 \theta)^{3/2} = \cos^3 \theta$$

$$\therefore \quad I = \int_0^{\pi/2} \sin^2 \theta \cos^3 \theta \cos \theta \, d\theta = \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta \, d\theta$$

Hence $I = \frac{(1)\cdot(3)(1)}{6\times4\times2}\cdot\frac{\pi}{2}$ by reduction formula.

Thus $I = \pi/32$

>> Let $I = \int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

Put $x = \sin^2\theta,\ dx = 2\sin\theta\ \cos\theta\ d\theta$ and $\theta$ varies from 0 to $\pi/2$.

Also $(1-x)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \sin^3\theta\cos^3\theta\cdot 2\sin\theta\cos\theta\,d\theta$$

*i.e.,* $$I = 2\int_0^{\pi/2}\sin^4\theta\cos^4\theta\,d\theta$$

Hence $I = 2\cdot\frac{[(3)(1)][(3)(1)]}{8\times6\times4\times2}\cdot\frac{\pi}{2}$ by reduction formula.

Thus $I = 3\pi/128$

>> Let $I = \int_0^1 \sin^{-1}x\cdot x^2\,dx$

Integrating by parts we get,

$$I = \left[\sin^{-1}x\cdot\frac{x^3}{3}\right]_0^1 - \int_0^1\frac{x^3}{3}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \left(\frac{\pi}{2}\cdot\frac{1}{3}-0\right)-\frac{1}{3}\int_0^1\frac{x^3}{\sqrt{1-x^2}}\,dx$$

Put $x = \sin\theta \quad \therefore\ dx = \cos\theta\,d\theta$ and $\theta$ varies from 0 to $\pi/2$

$$\therefore \quad I = \frac{\pi}{6} - \frac{1}{3}\int_{\theta=0}^{\pi/2} \frac{\sin^3\theta \cdot \cos\theta\, d\theta}{\cos\theta}$$

$$I = \frac{\pi}{6} - \frac{1}{3}\int_{0}^{\pi/2} \sin^3\theta\, d\theta$$

$$I = \frac{\pi}{6} - \frac{1}{3}\cdot\frac{2}{3} \text{ by applying reduction formula.}$$

Thus $I = \frac{1}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$

>> Let $I_1 = \int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$

Put $x = 2a\sin^2\theta \quad \therefore \quad dx = 4a\sin\theta\cos\theta\, d\theta$, $\theta$ varies from 0 to $\pi/2$

Also $\sqrt{2ax - x^2} = \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}$

*i.e.,* $= \sqrt{4a^2\sin^2\theta(1 - \sin^2\theta)} = \sqrt{4a^2\sin^2\theta\cos^2\theta} = 2a\sin\theta\cos\theta$

$$\therefore \quad I_1 = \int_{\theta=0}^{\pi/2} 4a^2\sin^4\theta \cdot 2a\sin\theta\cos\theta \cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= 32a^4\int_0^{\pi/2}\sin^6\theta\cos^2\theta\, d\theta$$

$$= 32a^4 \cdot \frac{[(5)(3)(1)][(1)]}{8\times 6\times 4\times 2}\cdot\frac{\pi}{2} \text{ by reduction formula.}$$

Thus $I_1 = 5\pi a^4/8$

(ii) $$I_2 = \int_0^{\pi/2} \frac{4a^2 \sin^4 \theta}{2a \sin \theta \cos \theta} \cdot 4a \sin \theta \cos \theta \, d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4 \theta \, d\theta$$

$$= 8a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ by reduction formula.}$$

Thus $\mathbf{I_2 = 3\pi a^2 / 2}$

>> Let $I = \int_0^a x\sqrt{ax - x^2}$

Put $x = a \sin^2 \theta \quad \therefore \quad dx = 2a \sin \theta \cos \theta \, d\theta, \quad \theta$ varies from 0 to $\pi/2$

Also $\sqrt{ax - x^2} = \sqrt{a^2 \sin^2 \theta - a^2 \sin^4 \theta} = \sqrt{a^2 \sin^2 \theta \, (1 - \sin^2 \theta)}$

$$= \sqrt{a^2 \sin^2 \theta \cos^2 \theta} = a \sin \theta \cos \theta$$

$\therefore$ $$I = \int_0^{\pi/2} a \sin^2 \theta \cdot a \sin \theta \cos \theta \cdot 2a \sin \theta \cos \theta \, d\theta$$

$$= 2a^3 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta$$

$$= 2a^3 \cdot \frac{(3 \cdot 1)(1)}{6 \times 4 \times 2} \cdot \frac{\pi}{2} \text{ by reduction formula.}$$

Thus $\mathbf{I = \pi a^3 / 16}$

>> Let $I = \int_{0}^{2a} x^n \sqrt{2ax - x^2}\, dx$

Put $x = 2a \sin^2 \theta \quad \therefore \quad dx = 4a \sin\theta \cos\theta \, d\theta$

$\theta$ varies from 0 to $\pi/2$. Also $\sqrt{2ax - x^2} = 2a \sin\theta \cos\theta$

$$\therefore \quad I = \int_{\theta = 0}^{\pi/2} (2a)^n \sin^{2n}\theta \cdot 2a \sin\theta \cos\theta \cdot 4a \sin\theta \cos\theta \, d\theta$$

$$= 2^{n+3} a^{n+2} \int_{0}^{\pi/2} \sin^{2n+2}\theta \cos^2\theta \, d\theta$$

Here $(2n + 2)$ is an even integer and hence the application of the reduction formula will give us

$$I = 2^{n+3} \, a^{n+2} \, \frac{[(2n+1)(2n-1)(2n-3)\cdots 1] \cdot 1}{(2n+4)(2n+2)(2n)\cdots 2} \cdot \frac{\pi}{2}$$

$$= 2^{n+2} \, a^{n+2} \cdot \frac{(2n+1)(2n-1)(2n-3)\cdots 1 \cdot \pi}{2(n+2)2(n+1)2n\,2(n-1)\cdots 2 \cdot 1}$$

$$= 2^{n+2} a^{n+2} \cdot \frac{(2n+1)(2n-1)(2n-3)\cdots 1 \cdot \pi}{2^{n+2}(n+2)!}$$

Multiplying both the numerator and the denominator by $2n(2n-2)(2n-4)\cdots 2$ in order to obtain $(2n+1)!$ in the numerator we have,

$$I = a^{n+2} \cdot \frac{(2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots 2.1 \cdot \pi}{(n+2)! \; 2n(2n-2)(2n-4)\cdots 2}$$

$$= \frac{a^{n+2} \cdot (2n+1)! \; \pi}{(n+2)! \; 2n \cdot 2(n-1) \cdot 2(n-2) \cdots 2.1}$$

$$= \frac{a^{n+2} \cdot (2n+1)! \; \pi}{(n+2)! \; 2^n \; n!}$$

Thus $\mathbf{I = \pi a^2 \left(\frac{a}{2}\right)^n \frac{(2n+1)!}{(n+2)!\, n!}}$ as required.

Now putting $n = 3$ we get,

$$I = \int_0^{2a} x^3 \sqrt{2ax - x^2}\, dx = \pi a^2 \left(\frac{a}{2}\right)^3 \frac{7!}{5!\,3!} \text{ by the above result.}$$

Thus $\quad I = \dfrac{\pi a^5 \cdot 7 \cdot 6\,(5!)}{8 \cdot 5! \cdot 3!} = \dfrac{7\pi a^5}{8}$

>> Let $\quad I = \displaystyle\int_0^1 \frac{x^9}{\sqrt{1-x^2}}\, dx$

Put $\quad x = \sin\theta \quad \therefore\ dx = \cos\theta\, d\theta$ and $\theta$ varies from $0$ to $\pi/2$.

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\sin^9\theta}{\cos\theta} \cos\theta\, d\theta = \int_0^{\pi/2} \sin^9\theta\, d\theta$$

Hence $I = \dfrac{8}{9} \cdot \dfrac{6}{7} \cdot \dfrac{4}{5} \cdot \dfrac{2}{3}$ by reduction formula.

Thus $\quad I = \mathbf{128 / 315}$

>> Let $I = \displaystyle\int_0^{\infty} \frac{x^4}{(1+x^2)^4}\, dx$

Put $\quad x = \tan\theta \quad \therefore\ dx = \sec^2\theta\, d\theta$

If $\quad x = 0,\ \theta = 0 \ ;\ $ If $\ x = \infty,\ \theta = \pi/2$

Also $\quad (1 + x^2)^4 = (1 + \tan^2\theta)^4 = (\sec^2\theta)^4 = \sec^8\theta$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\tan^4\theta}{\sec^8\theta} \sec^2\theta\, d\theta = \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^6\theta}\, d\theta$$

$$= \int_0^{\pi/2} \cos^6\theta \, \frac{\sin^4\theta}{\cos^4\theta} \, d\theta = \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta$$

Hence $I = \dfrac{(3)(1)\cdot(1)}{6\times4\times2}\cdot\dfrac{\pi}{2}$ by reduction formula.

Thus **$I = \pi/32$**

>> Put $x = \tan\theta$ $\therefore$ $dx = \sec^2\theta \, d\theta$ and $\theta$ varies from $0$ to $\pi/2$

Also $(1 + x^2)^{7/2} = (\sec^2\theta)^{7/2} = \sec^7\theta$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\tan^2\theta \cdot \sec^2\theta}{\sec^7\theta} \, d\theta = \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^5\theta} \, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^3\theta \, d\theta$$

Hence $I = \dfrac{(1)\cdot(2)}{5\times3\times1} = \dfrac{2}{15}$ by reduction formula.

Thus **$I = 2/15$**

>> Let $I = \displaystyle\int_0^{\infty} \frac{x^2}{(1+x^6)^{7/2}} \, dx$

Put $x^6 = \tan^2\theta$ *ie* $x = (\tan^2\theta)^{1/6} = \tan^{1/3}\theta$

$\therefore$ $dx = \dfrac{1}{3}\tan^{-2/3}\theta \sec^2\theta \, d\theta$ and $\theta$ varies from $0$ to $\pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\tan^{2/3}\theta}{\sec^7\theta} \cdot \frac{1}{3} \tan^{-2/3}\theta \sec^2\theta \, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^5\theta \, d\theta$$

$$= \frac{1}{3} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{45} \text{ by reduction formula.}$$

Thus $I = 8/45$

***Evaluate the following.***

1. $\int_0^{\pi} x \sin^5 x \, dx$

2. $\int_0^{\pi/6} \sin^5 3x \, dx$

3. $\int_0^{\pi/4} \cos^6 2x \, dx$

4. $\int_0^{\pi} x \sin^6 x \cos^4 x \, dx$

5. $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

6. $\int_0^{\pi/4} \sin^4 4x \cos^3 2x \, dx$

7. $\int_0^{\pi} x \sin^7 x \cos^2 x \, dx$

8. $\int_0^{\pi} \frac{\sqrt{1 - \cos x}}{1 + \cos x} \sin^2 x \, dx$

9. $\int_0^{1} x^5 \sin^{-1} x \, dx$

10. $\int_0^{a} x^2 (a^2 - x^2)^{3/2} \, dx$

11. $\int_0^{a} \frac{x^4}{\sqrt{a^2 - x^2}} \, dx$

12. $\int_0^{1} x^4 (1 - x^2)^{3/2} \, dx$

13. $\int_0^{1} \frac{x^3}{(1 + x^2)^4} \, dx$

14. $\int_0^{\infty} \frac{x}{(a^2 + x^2)^5} \, dx$

15. $\int_0^{\infty} \frac{x^6}{(1 + x^2)^{9/2}} \, dx$

| | | |
|---|---|---|
| 1. $8\pi/15$ | 2. $8/45$ | 3. $5\pi/64$ |
| 4. $3\pi^2/512$ | 5. $3\pi/128$ | 6. $128/1155$ |
| 7. $16\pi/315$ | 8. $8\sqrt{2}/3$ | 9. $11\pi/192$ |
| 10. $\pi a^6/32$ | 11. $3\pi a^4/16$ | 12. $3\pi/256$ |
| 13. $1/12$ | 14. $1/12a^6$ | 15. $1/7$ |

The equation $y = f(x)$ in the explicit form geometrically represents a curve. To draw this curve, the basic procedure is to take some values for $x$ and find the corresponding values of $y$. The set of points $(x, y)$ so tabulated are joined by a smooth curve if the points are non collinear. But we cannot employ this fundamental procedure if the equation of the curve is in the implicit form $f(x, y) = 0$ and it is complicated too.

This topic gives an insight to the process of finding the shape of a plane curve based on its equation by examining certain features. Based on these features we can draw a rough sketch of the curve. **It is highly essential to know the shape of the curve to find its area, length, surface area and volume of sôlids.**

1. *Symmetry* : If the given equation has even powers of $x$ only then the curve is symmetrical about the $y$-axis and if the given equation has even powers of $y$ only then the curve is symmetrical about the $x$-axis.

If $f(x, y) = f(y, x)$ then the curve is symmetrical about the line $y = x$. Also if $f(x, y) = f(-x, -y)$ then the curve is symmetrical about the origin.

2. *Special points on the curve* : If $f(0, 0) = 0$ then the curve passes through the origin. In such a case we can find the equations of the tangents at the origin by equating the groups of lowest degree terms in $x$ and $y$ to zero.

The points of intersection of the curve with the $x$-axis is got by putting $y = 0$ and that with the $y$-axis is got by putting $x = 0$.

3. *Asymptotes* : Asymptote of a given curve is defined to be the tangent to the given curve at infinity. In otherwords these are lines touching the curve at infinity. Equating the coefficient of highest degree terms in $x$ to zero we get asymptotes parallel to the $x$-axis and equating the coefficient of highest degree terms in $y$ to zero we get asymptotes parallel to the $y$-axis.

4. *Region of existence* : Region of existence can be determined by finding out the set of permissible (*real*) values of $x$ and $y$. The curve doesnot lie in the region whenever $x$ or $y$ is imaginary.

By examining these features we can draw a rough sketch of the curve.

**Note** : *In the case of a parametric curve : $x = x(t)$ and $y = y(t)$, we need to vary the parameter $t$ suitably to take a note of the variations in $x$ and $y$ so that the curve can be drawn accordingly.*

1. ***Symmetry*** **: If $f(r, \theta) = f(r, -\theta)$ then the curve is symmetrical about the initial line $\theta = 0$ and $\theta = \pi$.**

If $f(r, \theta) = f(r, \pi - \theta)$ then the curve is symmetrical about the line $\theta = \pi/2$ *(positive y- axis)*

If $f(r, \theta) = f(r, \pi/2 - \theta)$ then the curve is symmetrical about the line $\theta = \pi/4$ *(the line $y = x$)*

If $f(r, \theta) = f(r, 3\pi/2 - \theta)$ then the curve is symmetrical about the line $\theta = 3\pi/4$ *(the line $y = -x$)*

If $f(r, \theta) = f(-r, \theta)$ then the curve is symmetrical about the pole. *(origin)*

2. *Curve passing through the pole* : If $r = 0$ gives a single value of $\theta$ say $\theta_1$ between 0 and $2\pi$ then the curve passes through the pole once. $\theta = \theta_1$ is a tangent to the curve at the pole. If it gives two values then the curve passes through the pole twice.

3. *Asymptote* : If $r \to \infty$ as $\theta \to \theta_0$ then the line $\theta = \theta_0$ is an asymptote.

4. *Region of existence* : If $r$ is imaginary for $\theta \in (\alpha, \beta)$ *ie.,* $\alpha < \theta < \beta$ then the curve doesnot exist in the region between $\theta = \alpha$ and $\theta = \beta$.

5. *Special points* : We can tabulate a set of values of $r$ for convenient values of $\theta$. These give some specific points through which the curve passes.

By examining these features we can draw a rough sketch of the curve.

1.

>> We have $y^2 (a - x) = x^3$. [*This curve is known as cissoid.*]

We observe the following features of the curve.

1. *Symmetry* : The equation contain even powers of $y$.

$\Rightarrow$ the curve is symmetrical about the $x$- axis.

2. *Special points* : The curve passes through $(0, 0)$.

The given equation is $ay^2 - xy^2 = x^3$.

The lowest degree term is $ay^2$ and $ay^2 = 0 \Rightarrow y = 0$. which is the equation of the $x$-axis. Hence $x$-axis is the tangent to the curve at the origin.

Putting $y = 0$ we get $x = 0$ and vice- versa. This means that the curve meets the $x$-axis and $y$-axis at the origin only.

3. *Asymptotes* : Equating the coefficient of the highest degree term in $y$ *i.e* coefficient of $y^2$ being $a - x$ to zero we get $x = a$ which is a line parallel to the $y$-axis. Hence $x = a$ is an asymptote. Also coefficient of the highest degree term in $x$ is $x^3$ whose coefficient is 1. This implies that there is no asymptote parallel to the $x$-axis.

*Region of existence* : $y^2 = x^3/(a - x)$

$y = \sqrt{x^3/a - x}$. This is positive if $x > 0,\ a - x > 0$ or $x < 0,\ a - x < 0$ *i.e* $x > 0,\ x < a$ ; $x < 0,\ x > a$. Since $a > 0$ the second case is not possible. Hence $y$ is real if $x > 0$ and $x < a$ which implies that the curve lies in the interval $0 < x < a$. Further as $x$ increases $y$ also increases.

The shape of the curve is as follows.

**Note** : *Since the curve meets the coordinate axes at the origin only, the origin is called a 'cusp' with x-axis as the common tangent.*

>> $y^2 (a - x) = x^2 (a + x)$ [*This curve is known as 'Strophoid'.*]

We observe the following features of the curve.

1. *Symmetry* : The equation contain even powers of $y$.

$\Rightarrow$ the curve is symmetrical about the $x$-axis.

2. *Special points* : The curve passes through the origin. The equation of the curve can be put in the form

$$a(y^2 - x^2) - xy^2 - x^3 = 0.$$

Equating the lowest degree terms to zero we have $a(y^2 - x^2) = 0$

Hence $y = \pm x$ are the tangents to the curve at the origin. Since there are two tangents the origin is called a '*node*'.

Next, putting $y = 0$ we get $x^2(a + x) = 0 \Rightarrow x = 0, x = -a$.

The points are $(0, 0)$ and $(-a, 0)$

Also putting $x = 0$ we get $ay^2 = 0$ or $y = 0$ and the point is $(0, 0)$

Hence we say that the curve intersects the $x$-axis at $(0, 0)$ and $(-a, 0)$ and intersects the $y$- axis at $(0, 0)$ only.

3. *Asymptotes* : The coefficient of the highest degree in $x$ being $x^3$ is $-1$ and hence there is no asymptote parallel to the $x$-axis. Also the coefficient of the highest degree in $y$ being $a - x$, $a - x = 0$ gives $x = a$. Hence $x = a$ is the only asymptote which is a line parallel to the $y$-axis.

4. *Region of existence* : $y = \sqrt{x^2 (a+x)/a-x}$ .

When $a+x<0$ and $a-x>0$ ; $a+'x>0$ and $a-x<0$, $y$ is imaginary. Also when $a+x<0$ and $a-x<0$ $y$ is not imaginary.

Hence we can say that the curve lies between the lines $x = -a$ and $x = +a$ .
The shape of the curve is as follows.

>> We observe the following features of the curve.

1. *Symmetry* : The equation contain even powers of $y$ and hence the curve is symmetrical about the $x$ -axis.

2. *Special points* : The curve doesnot pass through the origin.

If $y = 0$ then $x = a$. The curve meets the $x$ -axis at $(a, 0)$ and it doesnot meet the $y$ -axis.

3. *Asymptotes* : The equation of the curve is $xy^2 - a^3 + a^2x = 0$. Coefficient of $y^2$ is $x$ and $x = 0$ being the $y$ -axis is an asymptote.
Also the coefficient of $x$ is $y^2 + a^2$ and $y^2 + a^2 = 0$ implies that $y$ is imaginary. Hence there is no asymptote parallel to the $x$ -axis.

4. *Region of existence* : $y^2 = a^2 (a-x)/x$ $\therefore$ $y = a\sqrt{(a-x)/x}$

$y$ is positive if $a-x>0$ and $x>0$ or $0<x<a$

Hence the curve lies between $x = 0$ and $x = a$.

The shape of the curve is as follows.

>> We observe the following features of the curve.

$f(r,\theta) \neq f(r,-\theta) \Rightarrow$ the curve is not symmetrical about the initial line.

$f(r,\theta) \neq f(-r,\theta) \Rightarrow$ the curve is not symmetrical about the pole.

$f(r,\theta) = f(r,\pi-\theta) \Rightarrow$ the curve is symmetrical about the line $\theta = \pi/2$.

$r = 0$ gives $\sin 3\theta = 0 \Rightarrow 3\theta = n\pi$ or $\theta = n\pi/3$

Taking values for $n = 0, 1, 2, \ldots 6$ we get the corresponding values of $\theta : 0$, $\pi/3$, $2\pi/3$, $\pi$, $4\pi/3$, $5\pi/3$, $2\pi$ and the curve passes through the pole for these values six times.

| | | | |
|---|---|---|---|
| If | $0 < \theta < \pi/6$, | $r$ is positive | and $r = a$ if $\theta = \pi/6$ |
| If | $\pi/6 < \theta < \pi/3$, | $r$ is positive | and $r = 0$ if $\theta = \pi/3$ |
| If | $\pi/3 < \theta < \pi/2$, | $r$ is negative | and $r = -a$ if $\theta = \pi/2$ |

These observations implies that $r$ increases from 0 to $a$ as $\theta$ varies from 0 to $\pi/6$, $r$ decreases from $a$ to 0 as $\theta$ varies from $\pi/6$ to $\pi/3$,

$r$ increases numerically from 0 to $a$ as $\theta$ varies from $\pi/3$ to $\pi/2$.

Further $f(r, \pi/3-\theta) = f(r,\theta)$ implies that the curve is symmetrical about the line $\theta = \pi/6$ so that we concIulde that there is a loop between the lines $\theta = 0$ and $\theta = \pi/3$.

Similarly we can examine the path of the curve as $\theta$ moves from $\pi/2$ to $\pi$ and also from $\pi$ to $2\pi$.

Let us tabulate a set of values of $r$ corresponding to some values of $\theta$

| $\theta^\circ$ | 0 | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r$ | 0 | $a$ | 0 | $-a$ | 0 | $a$ | 0 | $-a$ | 0 | $a$ | 0 | $-a$ | 0 |

The curve is symmetrical about $\theta = 5\pi/6$ and $3\pi/2$

The shape of the curve is as follows.

>> We observe the following features of the curve.

$f(r,\theta) = f(r,-\theta) \Rightarrow$ the curve is symmetrical about the initial line.
$f(r,\theta) = f(-r,\theta) \Rightarrow$ the curve is symmetrical about the pole.

$r = 0$ gives $a^2 \cos 2\theta = 0$

*ie.,* $\cos 2\theta = 0 \Rightarrow 2\theta = \pi/2$ and $3\pi/2$

$\therefore$ $\theta = \pi/4$ and $\theta = 3\pi/4$ are the tangents to the curve at the pole.

When $\theta = 0, r^2 = a^2$ or $r = \pm a$.

Hence the curve meets the initial line at the points $(+a, 0)$ and $(-a, 0)$.

Since the curve is symmetrical about the initial line it is composed of two loops. $r$ is real for $\theta \in [0, \pi/4]$ and $[3\pi/4, \pi]$. Also $r$ doesnot tend to infinity for any $\theta$ and hence there are no asymptotes.

The shape of the curve is as follows.

The results connected with the derivative of the arc length [ *Refer unit - II* ] will be useful in the discussion of finding the area, length / perimeter of plane curves, the surface area of the revolution of the curve about a given line. Further we also discuss the volume of a solid of revolution. The relevent formulae for finding these are as follows.

1. ***Area*** : The area $(A)$ bounded by $a$ curve $y = f(x)$, the $x$-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = \int_{x=a}^{b} y\,dx$$

The area $(A)$ between the curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

The area $(A)$ called the sectorial area bounded by a polar curve $r = f(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$

2. ***Length*** : The length of the arc of a curve between two specified points on it for various types of curves are given by the following formulae.

Such a process is called *rectification* and the entire length of the curve is called as the *perimeter* of the curve.

(i) *Cartesian curve* $y = f(x)$ *or* $x = f(y)$

$$s = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad \text{or} \quad \int_{y=c}^{d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

(ii) *Parametric curve* $x = x(t), y = y(t)$

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

(iii) *Polar curve* $r = f(\theta)$

$$s = \int_{\theta=\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{or} \quad s = \int_{r=r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr$$

3. ***Surface area*** : When a curve revolves about the $x$-axis a surface is generated and the same is called a surface of revolution. If a curve is bounded by the ordinates $x = a$ and $x = b$ revolves once completely about the $x$-axis, the area of the surface $(S)$ generated is given by

$$S = \int_{x=a}^{b} 2\pi y\, ds = \int_{a}^{b} 2\pi y \frac{ds}{dx}\, dx$$

where $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Similarly the surface area of revolution about the $y$ - axis is given by

$$S = \int_{y=c}^{d} 2\pi x\, ds = \int_{c}^{d} 2\pi x \frac{ds}{dy}\, dy,$$

where $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$

In the case of a polar curve the surface area of revolution about the initial line is given by

$$S = \int_{\theta=\theta_1}^{\theta_2} 2\pi r \sin\theta\, ds = 2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \frac{ds}{d\theta}\, d\theta$$

where $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

4. ***Volume of revolution*** : The volume $(V)$ of the solid generated by the revolution of the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$, about the $x$- axis is given by

$$V = \pi \int_{x=a}^{b} y^2\, dx$$

Similarly if the axis of revolution is the $y$-axis, the volume of the solid is given by

$$V = \pi \int_{y=c}^{d} x^2\, dy$$

Also in the case cf a polar curve $r = f(\theta)$ the volume $(V)$ of the solid generated is given by

$V = \dfrac{2\pi}{3} \int r^3 \sin\theta \; d\theta$ (*revolution about the initial line*)

$V = \dfrac{2\pi}{3} \int r^3 \cos\theta \; d\theta$ (*revolution about the line* $\theta = \pi/2$)

Applications formulae at a glance

| | Cartesian curve | Parametric curve | Polar curve |
|---|---|---|---|
| Area $(A)$ | $\int_a^b y \, dx$<br>or $\int_c^d x \, dy$ | $\int_{t_1}^{t_2} y \dfrac{dx}{dt} \, dt$<br>or $\int_{t_1}^{t_2} x \dfrac{dy}{dt} \, dt$ | $\dfrac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta$ |
| Length $(s)$ | $\int_a^b \dfrac{ds}{dx} \, dx$<br>or $\int_c^d \dfrac{ds}{dy} \, dy$ | $\int_{t_1}^{t_2} \dfrac{ds}{dt} \, dt$ | $\int_{\theta_1}^{\theta_2} \dfrac{ds}{d\theta} \, d\theta$<br>or $\int_{r_1}^{r_2} \dfrac{ds}{dr} \, dr$ |
| Surface area of revolution $(S)$ | $2\pi \int_a^b y \dfrac{ds}{dx} \, dx$<br>(about the $x$-axis)<br>$2\pi \int_c^d x \dfrac{ds}{dy} \, dy$<br>(about the $y$-axis) | $2\pi \int_{t_1}^{t_2} y \dfrac{ds}{dt} \, dt$<br>(about the $x$-axis)<br>$2\pi \int_{t_1}^{t_2} x \dfrac{ds}{dt} \, dt$<br>(about the $y$-axis) | $2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \dfrac{ds}{d\theta} \, d\theta$ |
| Volume of revolution $(V)$ | $\pi \int_a^b y^2 \, dx$<br>(about the $x$-axis)<br>$\pi \int_c^d x^2 \, dy$<br>(about the $y$-axis) | $\pi \int_{t_1}^{t_2} y^2 \dfrac{dx}{dt} \, dt$<br>(about the $x$-axis)<br>$\pi \int_{t_1}^{t_2} x^2 \dfrac{dy}{dt} \, dt$<br>(about the $y$-axis) | $\dfrac{2\pi}{3} \int r^3 \sin\theta \; d\theta$<br>(about the line $\theta = 0$ or $x$-axis)<br>$\dfrac{2\pi}{3} \int r^3 \cos\theta \; d\theta$<br>(about the line $\theta = \pi/2$ or $y$-axis) |

1. ***The Astroid*** : Astroid is the curve represented by the equation :

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Its parametric equation is $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

We shall find its shape first and then determine the associated area, perimeter, surface area and the volume.

We tabulate $x$, $y$ corresponding to certain angles of $\theta$ in the interval $[0, 2\pi]$.

| $\theta$ | $0$ | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|---|---|---|---|---|---|
| $x$ | $a$ | $0$ | $-a$ | $0$ | $a$ |
| $y$ | $0$ | $a$ | $0$ | $-a$ | $0$ |

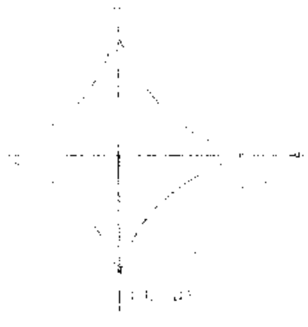
From the table we conclude that the curve meets the $x$-axis at the points $(a, 0)$ and $(-a, 0)$. Also it meets the $y$ - axis at the points $(0, a)$ and $(0, -a)$. Since $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$, we have $|x| \le a$ and $|y| \le a$. Hence we infer that the entire curve lies within a circle of radius '$a$' having origin as the centre.

Also we have from the cartesian equation of the curve,

$$f(x, y) = f(-x, y);\ f(x, y) = f(x, -y);\ f(x, y) = f(y, x)$$

Hence the curve is symmetrical about the coordinate axes and also about the line $y = x$.

Taking a note of the values of $x$ and $y$ as $\theta$ advances from one quadrant to the other the shape of the curve is as shown.



WORKED PROBLEMS

31 *Find the area* [illegible]

**Note** : *In any problem on applications we need to draw the curve first by briefly examining the important features.*

The curve astroid is symmetrical about the coordinate axes and hence the required area($A$) is equal to four times the area in the first quadrant.

*ie.,*
$$A = 4\int_0^a y\,dx = 4\int_0^a y\,\frac{dx}{d\theta}\,d\theta$$

We have $x = a\cos^3\theta,\ y = a\sin^3\theta \ \therefore\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\ \sin\theta$

When $x = 0 : a\cos^3\theta = 0$ or $\cos^3\theta = 0 \Rightarrow \theta = \pi/2$

$x = 1 : a\cos^3\theta = a$ or $\cos^3\theta = 1 \Rightarrow \theta = 0$

$$\therefore \quad A = 4\int_{\theta=\pi/2}^{0} a\sin^3\theta\,(-3a\cos^2\theta\ \sin\theta)\ d\theta$$

$$= 12a^2\int_0^{\pi/2}\sin^4\theta\ \cos^2\theta\ d\theta$$

$$= 12a^2\cdot\frac{(3)(1)\cdot(1)}{6\times4\times2}\cdot\frac{\pi}{2},\ \text{by reduction formula.}$$

**Thus the area enclosed** $(A)$ is $3\pi a^2/8$ sq. units.

>> Since the curve is symmetrical about the coordinate axes, the perimeter (*entire length*) of the curve is four times its length in the first quadrant.

$$\therefore \quad l = 4\int_{\theta=0}^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\ d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{9a^2\cos^4\theta\ \sin^2\theta + 9a^2\ \sin^4\theta\ \cos^2\theta}\ d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{9a^2\ \cos^2\theta\ \sin^2\theta(\cos^2\theta+\sin^2\theta)}\ d\theta$$

$$= 4\int_0^{\pi/2} 3a\cos\theta\ \sin\theta\ d\theta$$

$$= 6a\int_0^{\pi/2}\sin 2\theta\ d\theta$$

$$= 6a\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = -3a(\cos\pi-\cos 0) = -3a(-1-1) = 6a$$

**Thus the perimeter of the curve is $6a$ units.**

33. Find the surface [illegible] about the x-axis.

>> Because of symmetry the required surface area is equal to twice the surface area by the revolution of the first quadrant of the curve.

$$\therefore \qquad S = 2 \times \int_{0}^{a} 2\pi y \, ds = 4\pi \int_{\theta=0}^{\pi/2} y \frac{ds}{d\theta} \, d\theta$$

$$\text{But} \quad \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 3a \cos\theta \sin\theta \cdots \textit{(Refer Ex - 32)}$$

$$\text{Hence } S = 4\pi \int_{0}^{\pi/2} a \sin^3\theta \cdot 3a \cos\theta \sin\theta \, d\theta$$

$$= 12\pi a^2 \int_{0}^{\pi/2} \sin^4\theta \cos\theta \, d\theta$$

$$= 12\pi a^2 \cdot \frac{(3)}{5 \times 3 \times 1} \text{ by reduction formula.}$$

**Thus the required surface area = $12\pi a^2/5$ sq.units.**

---

34. Find the volume of the solid [illegible] the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

>> Because of symmetry the required volume ($V$) is equal to twice the volume of the solid generated by the curve in the first quadrant about the $x$-axis.

$$\therefore \qquad V = 2 \times \int_{0}^{a} \pi y^2 \, dx = 2\pi \int_{0}^{a} y^2 \frac{dx}{d\theta} \, d\theta$$

$$= 2\pi \int_{\theta=\pi/2}^{0} a^2 \sin^6\theta \, (-3a \cos^2\theta \sin\theta) \, d\theta$$

$$= 6\pi a^3 \int_{0}^{\pi/2} \sin^7\theta \cos^2\theta \, d\theta$$

$$= 6\pi a^3 \frac{(6)(4)(2)\cdot(1)}{9 \times 7 \times 5 \times 3 \times 1} \text{ by reduction formula.}$$

**Thus the required volume of the solid is $32\pi a^3/105$ cubic units.**

---

**2.** ***Cycloid*** : Cycloid is a curve generated by a point on the circumference of a circle which rolls on a fixed straight line known as the base. Imagine a wheel rolling on a straight line without slipping. A fixed point on the rim of the wheel traces the cycloid. The parametric equation of the cycloid can be in the following four forms :

(i) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

(ii) $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$

(iii) $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

(iv) $x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta)$

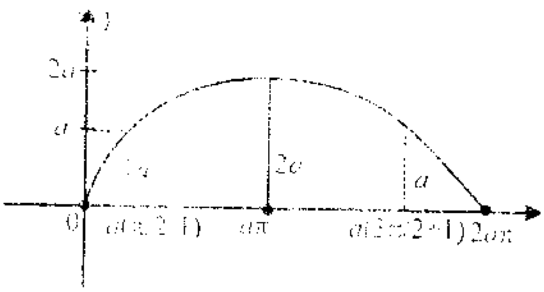
*Tracing of the cycloid :* $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$.

>> Let us tabulate $x,\ y$ for certain values of $\theta$ in the interval $[0,\ 2\pi]$ where $\theta$ is in radians.

| $\theta$ | $0$ | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|---|---|---|---|---|---|
| $x$ | $0$ | $a(\pi/2 - 1)$ | $a\pi$ | $a(3\pi/2 + 1)$ | $2a\pi$ |
| $y$ | $0$ | $a$ | $2a$ | $a$ | $0$ |

From the table we can conclude that the curve intersects the $x$ - axis at $x = 0$ and $2a\pi$. Also, we have $y = a(1 - \cos\theta)$ and since $|\cos\theta| \le 1$ $y$, is non negative. Hence the curve lies above the $x$-axis.

Taking a note of the values of $x$ and $y$ as $\theta$ advances in the interval $[0,\ 2\pi]$ the shape of the curve is as follows. It is called an arch of the curve.




## WORKED PROBLEMS

35. *Find the area of an arch of the cycloid* $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$.

>> Area $$A = \int_{\theta=0}^{2\pi} y \frac{dx}{d\theta}\, d\theta$$

*ie.,* $$A = \int_0^{2\pi} a(1 - \cos\theta) \cdot a(1 - \cos\theta)\, d\theta$$

$$= a^2 \int_0^{2\pi} (1 - \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{2\pi} 4\sin^4(\theta/2) \, d\theta$$

Put $\theta/2 = t \quad \therefore \; d\theta = 2dt,\ t$ varies from 0 to $\pi$

$$\therefore \quad A = 8a^2 \int_{t=0}^{\pi} \sin^4 t \, dt = 8a^2 \cdot 2 \int_0^{\pi/2} \sin^4 t \, dt$$

*ie.,* $A = 16a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ by reduction formula.

**Thus the area enclosed by an arch of the curve on its base is $3\pi a^2$ sq.units**

>> Length $(l) = \int_{\theta=0}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$

*ie.,* $l = \int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2 \sin^2\theta} \, d\theta$

$$= \int_0^{2\pi} a \sqrt{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} \, d\theta$$

$$= a \int_0^{2\pi} \sqrt{2(1-\cos\theta)} \, d\theta = a \int_0^{2\pi} \sqrt{2 \cdot 2 \sin^2(\theta/2)} \, d\theta$$

$$= 2a \int_0^{2\pi} \sin(\theta/2) \, d\theta = -\left[ \frac{2a \cos(\theta/2)}{1/2} \right]_0^{2\pi}$$

$$= -4a(\cos\pi - \cos 0) = -4a(-1-1) = 8a$$

**Thus the required length is $8a$**

>> Surface area $S = 2\pi \int_0^{2\pi} y \frac{ds}{d\theta}\, d\theta$

But $\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2a \sin(\theta/2)$ [*Refer Ex-36*]

$\therefore \quad S = 2\pi \int_0^{2\pi} a(1 - \cos\theta) \cdot 2a \sin(\theta/2)\, d\theta$

$= 4\pi a^2 \int_0^{2\pi} 2 \sin^3(\theta/2)\, d\theta = 8\pi a^2 \int_0^{2\pi} \sin^3(\theta/2)\, d\theta$

Put $\theta/2 = t \quad \therefore\ d\theta = 2dt$ and $t$ varies from $0$ to $\pi$.

Hence $S = 8\pi a^2 \int_{t=0}^{\pi} \sin^3 t \cdot 2dt = 16\pi a^2 \cdot 2 \int_0^{\pi/2} \sin^3 t\, dt$

*ie.,* $S = 32\pi a^2 \cdot \frac{2}{3}$ by applying reduction formula.

**Thus the required surface area is $64\pi a^2/3$ sq.units.**

>> $V = \pi \int_0^{2\pi} y^2 \frac{dx}{d\theta}\, d\theta$

$= \pi \int_0^{2\pi} a^2 (1 - \cos\theta)^2 \cdot a(1 - \cos\theta)\, d\theta$

$= \pi a^3 \int_0^{2\pi} \left\{2\sin^2(\theta/2)\right\}^3 d\theta = 8\pi a^3 \int_0^{2\pi} \sin^6(\theta/2)\, d\theta$

$= 8\pi a^3 \cdot 2 \int_0^{\pi} \sin^6(\theta/2)\, d\theta = 16\pi a^3 \int_0^{\pi} \sin^6(\theta/2)\, d\theta$

Put $\theta/2 = t \therefore d\theta = 2dt$ and $t$ varies from $0$ to $\pi/2$

$$\therefore \quad V = 16\pi a^3 \int_0^{\pi/2} \sin^6 t \cdot 2\,dt$$

$$= 32\pi a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ by reduction formula.}$$

**Thus the required volume is $5\pi^2 a^3$ cubic units.**

### 3. *Cardioide* :

*Tracing of the cardioide* [illegible]

**We observe the following features of the curve.**

(i) $f(r, \theta) = f(r, -$ [illegible] about the initial [illegible]

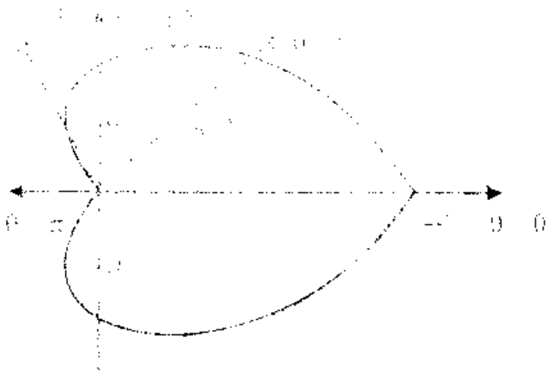
(ii) When $\theta = \pi$, $r =$ [illegible] and [illegible] curve pa[illegible] through the pole [illegible] tangent to the curve at [illegible]

(iii) Since $|\cos\theta| \leq$ [illegible] within the circle of [illegible] $2a$ having its centre at [illegible]

**Let us tabulate $r$ for certain angles of $\theta$**

| $\theta$ | $0$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $2a$ | $3a/2$ | $a$ | $a/2$ | $0$ |

It is evident that as $\theta$ increases from $0$ to $\pi$, $r$ decreases from $2a$ to $0$. The shape of the curve is as follows.



>> Since the curve is symmetrical about the initial line, the total area ( $A$ ) is twice the area above the initial line.

ie., $$A = 2 \cdot \frac{1}{2} \int_0^{\pi} r^2 \, d\theta = \int_0^{\pi} a^2 (1 + \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} [\, 2\cos^2(\theta/2)\,]^2 \, d\theta = 4a^2 \int_0^{\pi} \cos^4(\theta/2) \, d\theta$$

Put $\theta/2 = t$ $\quad \therefore \; d\theta = 2dt$ and $t$ varies from $0$ to $\pi/2$

$$\therefore \quad A = 4a^2 \int_{t=0}^{\pi/2} \cos^4 t \cdot 2dt = 8a^2 \int_0^{\pi/2} \cos^4 t \, dt$$

$$= 8a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$ , by reduction formula.

**Thus the area enclosed is $3\pi a^2/2$ sq.units.**

---

>> Perimeter (*length*) = 2 (length of the upper half of the curve)

ie., $$= 2 \int_0^{\pi} \frac{ds}{d\theta} \, d\theta \quad \text{where} \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Now $$\frac{ds}{d\theta} = \sqrt{a^2 (1 + \cos\theta)^2 + a^2 \sin^2\theta} = a\sqrt{2(1 + \cos\theta)}$$

$$= 2a \cos(\theta/2)$$

$$\therefore \quad \text{perimeter} = 2 \int_0^{\pi} 2a \cos(\theta/2) \, d\theta = 4a \left[ \frac{\sin(\theta/2)}{1/2} \right]_0^{\pi} = 8a$$

**Thus the perimeter of the curve is $8a$ units.**

---

>> Surface area $S = \int 2\pi r \sin\theta \frac{ds}{d\theta}\, d\theta$

But $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2a\cos(\theta/2)$ *(Refer Example - 40)*

$\therefore \quad S = 2\pi \int_{\theta=0}^{\pi} a(1+\cos\theta)\sin\theta \cdot 2a\cos(\theta/2)\, d\theta$

$= 4\pi a^2 \int_0^{\pi} 2\cos^2(\theta/2)\cdot 2\sin(\theta/2)\cos(\theta/2)\cos(\theta/2)\, d\theta$

$= 16\pi a^2 \int_0^{\pi} \cos^4(\theta/2)\sin(\theta/2)\, d\theta$

Put $\theta/2 = t \quad \therefore \quad d\theta = 2dt$ and $t$ varies from 0 to $\pi/2$

Hence $S = 16\pi a^2 \int_{t=0}^{\pi/2} \cos^4 t \sin t \cdot 2\, dt$

$= 32\pi a^2 \cdot \frac{(3)(1)}{5\times 3}$ by reduction formula.

**Thus the required surface area is $32\pi a^2/5$ sq.units.**

>> $V = \frac{2}{3}\pi \int_0^{\pi} r^3 \sin\theta\, d\theta = \frac{2\pi}{3}\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta\, d\theta$

Put $t = 1+\cos\theta \quad \therefore \quad dt = -\sin\theta\, d\theta$

If $\theta = 0,\ t = 2$ and if $\theta = \pi,\ t = 0$

$\therefore \quad V = \frac{2\pi a^3}{3}\int_2^0 t^3(-dt) = \frac{2\pi a^3}{3}\int_0^2 t^3\, dt$

$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 = \frac{2\pi a^3}{3}(4-0) = \frac{8\pi a^3}{3}$

**Thus the required volume is $8\pi a^3/3$ cubic units.**